From number theory to physics: Introducing η regularisation

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Based on [2401.10981] in collaboration with Tony Padilla.

Some examples of ongoing and future research directions

Motivations: UV finiteness of string amplitudes, and the longstanding hope quantum gravity will act as a universal regulator in the lower-energy limit of QFT.

String theory

- String amplitudes
  - Gross-Mende
  - Handle operators
  - Analytic and/or non-analytic structure of UV divergences
    - η regularisation in moduli space
- Schwinger representation and worldline formalism
- String species scale and smooth asymptotics

Quantum field theory

- η regularisation as a general symmetry preserving method
  - Gauge theory and master equations
  - Broader structure of η regularisation + relation between different η's
    - Capturing all common regularisation methods
  - Anomalies
  - Curved spacetime
  - Generalisation to n-loops

Analytic number theory:

- Divergent series and smooth asymptotics
  - η class of smooth cutoff functions and generalised functions
  - Riemann zeta and extension to non-integer s for R(s)>1.
  - Asymptotics of distributions
    - Quasiasymptotics, Tauberian theorems, etc.
  - Laplace transform and other interesting relations
- Modular forms and L-series
- Resurgence theory

Sums of integer powers

Consider the following sums of integer powers written in terms of their partial sums:

Examples in physics tells us that the sum of the naturals should be attributed the value -1/12.

$$1+2+3+\ldots+n=\frac{1}{2}n(n+1)=\frac{1}{2}n+\frac{1}{2}n^2,$$

$$1^2+2^2+3^2+\ldots+n^2=\frac{1}{6}n(n+1)(2n+1)=\frac{1}{6}n+\frac{1}{2}n^2+\frac{1}{3}n^3,$$

$$1^3+2^3+3^3+\ldots+n^3=\frac{1}{4}n^2(n+1)^2=\frac{1}{4}n^2+\frac{1}{2}n^3+\frac{1}{4}n^4.$$

Classically, when analysing infinite series we might consider the definition of divergence in the Cauchy sense: 1) a series that grows in absolute value without limit or, 2) a series that is bounded but whose sequence of partial sums does not approximate any specific value.

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n.$$

Divergent series

Compare the partial sums with the values given by the Riemann zeta function

Tao, 2011

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \dots.$$

More generally, after analytic continuation: $\zeta(-s) = -\frac{B_{s+1}}{s+1}$

Formally applying values s = 1,2,… one will obtain:

$$\sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + \dots = -1/2$$

$$\sum_{n=1}^{\infty} n = 1 + 2 + 3 + \dots = -1/12,$$

$$\sum_{n=1}^{\infty} n^2 = 1 + 4 + 9 = 0.$$

$$\sum_{n=1}^{\infty} n^s = 1^s + 2^s + 3^s + \dots = -\frac{B_{s+1}}{s+1}$$

Much has been said of these rather bizarre, if not altogether absurd, formulae. They were of course also made famous by Ramanujan.

- They do not appear coherent or reasonable because, as written, these formulae are characterised by positive summands on the left-hand side appearing to equate to some negative or zero value.

- Comparing with the previous partial sums, one can try to inspect the partial sums of these divergent series. But there is no obvious relationship with these constant values.

Compare divergent series and partial sums

Tao, 2011

Define $S_s(N) = \sum_{k=1}^{N} k^s$ then the previous partial sums can be expressed as special cases of Faulhaber's formula

$$S_s(N) = \frac{1}{s+1}\sum_{k=0}^{s}\binom{s+1}{k} B_k N^{s+1-k}$$ where B_k denotes the Bernoulli numbers.

$$= \frac{1}{s+1}N^{s+1} + \frac{1}{2}N^s + \frac{s}{12}N^{s-1} + \ldots + B_s N$$

In the limit N -> ∞ Faulhaber's formula breaks down. Quite simply, this is because in the Cauchy sense an infinite series is divergent.

Comparing the partial sums behaviour with the divergent series, Terence Tao shows: if N is considered a real number, then this sum has jump discontinuities (of the first kind) at each positive integer value of N.

- In short, when we use partial sums to sum an infinite series, we truncate the series at some finite value N.
  - And so we can think of the partial sum as modifying the infinite series with a step function

$$\sum_{n=1}^{\infty} a_n \to \sum_{n=1}^{\infty} a_n \theta\left(\frac{n}{N}\right) = \sum_{n=1}^{N} a_n.$$

$$\theta(x) = \begin{cases} 1 & x \le 1 \\ 0 & x > 1 \end{cases}$$

- In the traditional partial sums, the discontinuities that produce various artefacts arise due to discretisation as a result of the abrupt truncation of the sum at some N.

Tao's method of smooth summation

Tao, 2011

Instead, as Tao motivates, we can consider smooth sums of the form $\sum_{n=1}^{\infty} \eta\left(\frac{n}{N}\right) a_n$

in which the notion of convergence is now defined as $\lim_{N\to\infty} \sum_{n=1}^{\infty} a_n \eta\left(\frac{n}{N}\right) = s.$

We define a smooth cutoff function η(x) : R+ → R that is a bounded function of compact support, which is taken to be the interval [0,1]. This means that x → ∞ η(x) = 0 and η(0) = 1.

This leads to the generalised Euler-Maclaurin formula

$$\int_0^N f(x)\eta(x)\, dx = \frac{1}{2} f(0) + \sum_{k=1}^{N} f(k)\eta(k) + \sum_{i=2}^{s+1} \frac{B_i}{i!} f^{(i-1)}(0) + O(N \mid\mid f^{(s+2)} \mid\mid_\infty),$$

where $\mid\mid f^{(s+2)} \mid\mid_\infty = \sup_{x\in\mathbb{R}} \mid f^{(s+2)(x)} \mid.$

Relation to the zeta function

From the generalised Euler-Maclaurin formula, smooth summation can be related to the Riemann zeta function:

Tao, 2011

$$\sum_{n=0}^{\infty} n^s \, \eta\left(\frac{n}{N}\right) = C_{\eta,s} N^{s+1} + \zeta(-s) + O\left(\frac{1}{N}\right),$$

where $C_{\eta,s} N^{s+1} = N^{s+1} \int_0^\infty dx \; x^s \; \eta(x)$ is the Mellin transform of the smooth cutoff.

The overall divergent sum can be decomposed into a finite part and an infinite part. The finite piece doesn't depend on the choice of regulator.

This is true if η(x) is a sufficiently smooth function.

Notice: the divergent pieces of the smooth sums scale like $C_{\eta,s} N^{(s+1)}$ which is scheme dependent.

So for the correct choice of cutoff the integral can be killed completely, thus exposing the unique finite value of the divergent series as perhaps the most natural value.

Example: Sums of powers

Consider again the sums of integer powers. Their smooth asymptotics give:

$$\sum_{n=1}^{\infty} \eta\left(\frac{n}{N}\right) = -\frac{1}{2} + C_{\eta,0} N + O\left(\frac{1}{N}\right),$$

$$\sum_{n=1}^{\infty} n\,\eta\left(\frac{n}{N}\right) = -\frac{1}{12} + C_{\eta,1} N^2 + O\left(\frac{1}{N}\right),$$

$$\sum_{n=1}^{\infty} n^2\,\eta\left(\frac{n}{N}\right) = 0 + C_{\eta,2} N^3 + O\left(\frac{1}{N}\right).$$

The task to regularise any divergent series essentially reduces to the appropriate choice of regulator that kills the integral

$$C_{\eta,s} N^{s+1} = N^{s+1} \int_0^{\infty} dx\; x^s\; \eta(x)$$

For example, for s = 0, we may equivalently write

$$\lim_{N\to\infty}\left[\sum_{n=1}^{\infty} \eta\left(\frac{n}{N}\right) - C_{\eta,0}\, N\right] = -\frac{1}{2}$$ which, after killing the integral, gives $$\sum_{n=1}^{\infty} 1 = -\frac{1}{2}$$

where the value -1/2 is revealed as the cutoff independent part of the sum.

Extending Tao's method

We show that many of the key ideas of Tao's method can be extended.

One important extension, we show (with proof) that the smooth regulating function η can be extended to the much more general class of Schwartz functions.

$$\mathcal{S}(\mathbb{R}^n, \mathbb{C}) := \left\{ f \in C^\infty(\mathbb{R}^n, \mathbb{C}) \mid \forall \alpha, \beta \in \mathbb{N}^n, \| f \|_{\alpha,\beta} < \infty \right\}$$

where $\| f \|_{\alpha,\beta} := \sup_{x \in \mathbb{R}^n} | x^\alpha (D^\beta f)(x) |$

$\eta : \mathbb{R} \mapsto \mathbb{R}$ is defined to be a Schwartz function if for every r ≥ 0 the rth derivative exists and goes rapidly to zero.

Example: any function that displays exponential decay at infinity will typically fall into the Schwartz class.

Even more generally, the class of smooth cutoff functions η can be any C^∞ function so long that we can ensure η(0)=1.

Another extension: Polynomial series

Consider the infinite sum $\sum_{k=1}^{\infty} f(k)$ where $f(k) = \sum_{s=0}^{z} c_s k^s$ is a polynomial of degree z.

From the generalised Euler-Maclaurin formula in which we define $g_N(x) = f(x)\eta\left(\frac{x}{N}\right)$ we show with proof

$$\sum_{k=1}^{\infty} f(k)\eta\left(\frac{k}{N}\right) = \sum_{s=0}^{z} c_s[C_{s,\eta}N^{s+1} + \zeta(-s)] + O\left(\frac{1}{N}\right)$$

Due to extending $\eta(x)$ to be Schwartz, it can be shown crucially that the integral in the remainder term of the Euler-Maclaurin formula is bounded and goes like O(1/N).

Importantly, we see $C_{s,\eta} = \int_0^{\infty} dx\ x^s \eta(x)$ is the Mellin transform of the smooth regulator function.

Observation: We find in general that power law divergences are regulator dependent and weighted by the corresponding Mellin transform.

- This is a feature reminiscent of QFT.

- Interestingly, the regulator dependence in the above raises the possibility that there are families of enhanced regulators for which the divergences vanish altogether!

Enhanced regulators

Definition: an *enhanced regulator* is one for which the Mellin transform $C_{s,\eta} = \int_0^\infty dx\ x^s \eta(x)$ vanishes for integer values of s ≥ 0.

An extremely elegant example of an enhanced regulator of order s is given by

$$\eta_s(x) = e^{-x\cot\left(\frac{\frac{\pi}{2}-\theta}{s+1}\right)} \frac{\cos(x+\theta)}{\cos\theta}$$

where $0 < \theta < \frac{\pi}{2}$ and s is any natural number.

For θ = 0 and s = 1 we recover the astonishingly beautiful enhanced regulator of order one $\eta_1(x) = e^{-x}\cos x$

from which it can be inferred $\lim_{N\to\infty} \sum_{n=1}^{\infty} n e^{-\frac{n}{N}} \cos\left(\frac{n}{N}\right) = -\frac{1}{12}$ for the sum of natural numbers.

Note: We also define *super-enhanced regulators* for polynomial series, and for both monomial and polynomial cases we have defined an algorithm for finding enhanced and super-regulators of any given order given for any Schwartz function.

Generalising to QFT: Divergent integrals at one-loop

We're motivated to ask an interesting question: can the method of smoothed asymptotics be extended to divergent Feynman integrals of quantum field theory?

For infinite series, smoothed asymptotics amounts to replacing an infinite sum with a smooth regulating function. This essentially means we replace an infinite sum with a smoothly weighted infinite sum

$$\sum_{n=0}^{\infty} \# \to \sum_{n=0}^{\infty} \eta\left(\frac{n}{N}\right) \#$$

The generalisation of this to divergent loop integrals is straightforward: working in Euclidean signature, we replace the loop integral with a smoothly weighted loop integral

$$\int d^4k \# \to \int d^4k \; \eta\left(\frac{|k|}{\Lambda}\right) \#,$$ where $|k|$ is the norm of the Euclidean four-momentum and Λ is the cut-off scale.

This represents a second important extension to Tao's method: we observe that the regulator cutoff may be interpreted as entering via a modification of the integration measure.

- From a physics point of view, this interpretation shares the philosophy of dimensional regularisation and is mathematically similar to the concept of mollification.

In our first paper [2401.10981] we make the simplest choice for η(x) such that x = k/Λ. More general choices are possible, and we start to explore them in our second paper (forthcoming).

Generalising to QFTs: Naïve example at one-loop

Example at one-loop

$$\mathcal{M} = \frac{(i\lambda)^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2} \frac{i}{(k+p)^2 - m^2}$$

After Feynman parameterisation and Wick rotating to Euclidean space

$$I = \frac{(-i\lambda)^2}{2} \int_0^1 dx \int d^4k \frac{-i}{(k^2 + M^2)^2}.$$

We can now follow our regularisation procedure by including a smooth cutoff

$$I^R = \frac{(-i\lambda)^2}{2} \int_0^1 dx \int d^4k \frac{-i}{(k^2 + M^2)^2} \eta(\epsilon k)$$

Using change of variables and some minor computation we have two integrals

$$I^R = \frac{-i^3\lambda^2\epsilon^{-2}}{2} \int_0^1 dx \int dq\; q\; \eta(q) - M^2 \int dq\; \frac{1}{q}\eta(q) \sim -M^2 \log(q) + \text{higher order corrections.}$$

The naïve observation is that we can regularise any divergent Feynman integral at one-loop.

- It is generally easy to regularise divergent integrals. The hard part is to do it consistently and for all QFTs.
- Any regularisation prescription worth its salt should satisfy locality, causality, and the Ward identity.
- So the question is, how do we formalise these naïve observations?

Irreducible loop integrals (one-fold ILIs) at one-loop

In order to define a consistent and useful regularisation prescription, we utilise the concept of irreducible loop integrals (ILIs) as first introduced in [Wu, 2002].

In general, the set of ILIs can be written as the following master integrals:

$$I_{-2\alpha} = \int d^4k \frac{1}{(k^2 - M^2)^{2+\alpha}}$$

$$I_{-2\alpha\,\mu\nu} = \int d^4k \frac{k_\mu k_\nu}{(k^2 - M^2)^{3+\alpha}}$$

$$I_{-2\alpha\,\mu\nu\rho\sigma} = \int d^4k \frac{k_\mu k_\nu k_\rho k_\sigma}{(k^2 - M^2)^{4+\alpha}}$$

- The mass factor M is a function of Feynman parameters, external momenta, and corresponding mass scales.
- The number (-2α) in the subscript labels the power counting dimension (of energy-momentum).
  - Here α = -1,0,1,2,… and, for the case where α = 0 and α = -1, one obtains the corresponding logarithmically and quadratically divergence integrals at one-loop, respectively.
- All one-loop Feynman integrals can be reduced to their respective ILIs.
- The concept of ILIs can be generalised to arbitrary loop order.

Regularised ILIs at one-loop

To regularise any divergent Feynman integral at one-loop, we must first apply Feynman parametrisation (sometimes repeatedly) to the amplitude as a whole, then reduce the integral to the appropriate ILIs.

Given that all one-loop integrals can be expressed in terms of the one-fold ILIs by way of the Feynman parameter method, we see that the divergences are thus completely characterized by these master integrals.

For example, we have the following master set of regularised integrals (in 4-dimensional Euclidean space) as related to the set of ILIs at one-loop:

$$I_2^R = -iJ_2^R[\eta_2] = -i\int d^4k\, \frac{1}{k^2+M^2}\eta_2\left(\frac{|k|}{\Lambda}\right)$$

$$I_0^R = iJ_0^R[\eta_0] = i\int d^4k\, \frac{1}{(k^2+M^2)^2}\eta_0\left(\frac{|k|}{\Lambda}\right)$$

$$I_{2\mu\nu}^R = -iJ_{2\mu\nu}^R[\tilde{\eta}_2] = -i\int d^4k\, \frac{k^2}{(k^2+M^2)^2}\tilde{\eta}_2\left(\frac{|k|}{\Lambda}\right)$$

Regularised ILIs at one-loop: Decomposition and asymptotics

Making use of partial fraction decomposition, the regularised tensor ILIs can be written explicitly in terms of scalar counterparts

$$J^{\mu\nu}_{-2\alpha}[\eta](\mathcal{M}^2) = \frac{1}{4} g^{\mu\nu} \left[J_{-2\alpha}[\eta](\mathcal{M}^2) - \mathcal{M}^2 J_{-2(\alpha+1)}[\eta](\mathcal{M}^2) \right] ,$$

$$J^{\mu\nu\rho\sigma}_{-2\alpha}[\eta](\mathcal{M}^2) = \frac{1}{4!} S^{\mu\nu\rho\sigma} \left[J_{-2\alpha}[\eta](\mathcal{M}^2) - 2\mathcal{M}^2 J_{-2(\alpha+1)}[\eta](\mathcal{M}^2) + \mathcal{M}^4 J_{-2(\alpha+2)}[\eta](\mathcal{M}^2) \right]$$

And so, it suffices to study the η regulated scalar ILIs and the structure of the divergences for different values of α.

- Most interesting case is for $\alpha \leq 0$, where the integrals diverge as $\Lambda \to \infty$ and take the following asymptotic form

$$J_0[\eta](\mathcal{M}^2) \sim \frac{1}{8\pi^2} \left[\ln(\Lambda/|\mathcal{M}|) + \gamma[\eta] - \frac{1}{2} \right] ,$$

$$J_2[\eta](\mathcal{M}^2) \sim \frac{1}{8\pi^2} \left[\Lambda^2 C_1[\eta] - \mathcal{M}^2 \left(\ln(\Lambda/|\mathcal{M}|) + \gamma[\eta] \right) \right] ,$$

and

$$J_{2s+4}[\eta](\mathcal{M}^2) \sim \frac{1}{8\pi^2} \left[\sum_{z=0}^{s} \binom{s}{z} C_{2z+3}[\eta] \mathcal{M}^{2(s-z)} \Lambda^{2z+4} \right]$$

Here s is any natural number.

Gauge theory (one-loop): Vacuum polarisation and Ward identities

For general non-Abelian gauge theory can write the vacuum polarisation function $\Pi_{\mu\nu}(k) = k_\mu k_\nu \Pi(k^2) - g_{\mu\nu}k^2\Pi(k^2)$

which we can then write in terms of ILIs $= \chi \int_0^1 dx \left[2I_{2\mu\nu}(m) - I_2(m) g_{\mu\nu} + 2x(1-x)(p^2 g_{\mu\nu} - p_\mu p_\nu) I_0(m) \right]$

where terms proportional to p satisfy the Ward identity $p^\mu \Pi^{ab}_{\mu\nu} = \Pi^{ab}_{\mu\nu} p^\nu = 0.$

From this analysis we are led to the following gauge consistency conditions in the asymptotic limit

$$I^{\mu\nu}_{-2\alpha}\big|_{\text{regularised}} \sim \frac{1}{2(\alpha+2)} g^{\mu\nu} I_{-2\alpha}\big|_{\text{regularised}},$$
$$I^{\mu\nu\rho\sigma}_{-2\alpha}\big|_{\text{regularised}} \sim \frac{1}{4(\alpha+2)(\alpha+3)} S^{\mu\nu\rho\sigma} I_{-2\alpha}\big|_{\text{regularised}},$$

We can apply these conditions in η regularisation using identities like $I_{-2\alpha}\big|_{\text{regularised}} = iJ_{-2\alpha}[\eta_{-2\alpha}]$ from a previous slide as well as the decomposition formulae

$$\frac{\alpha}{\alpha+2} J_{-2\alpha}\left[\tilde{\theta}_{-2\alpha}\right] \sim \mathcal{M}^2 J_{-2(\alpha+1)}[\theta_{-2\alpha}], \qquad \tilde{\theta}_{-2\alpha} = \frac{(\alpha+2)\theta_{-2\alpha} - 2\eta_{-2\alpha}}{\alpha},$$

$$\frac{\alpha(\alpha+5)}{(\alpha+2)(\alpha+3)} J_{-2\alpha}\left[\tilde{\kappa}_{-2\alpha}\right] \sim 2\mathcal{M}^2 J_{-2(\alpha+1)}[\kappa_{-2\alpha}] - \mathcal{M}^4 J_{-2(\alpha+2)}[\kappa_{-2\alpha}], \qquad \tilde{\kappa}_{-2\alpha} = \frac{(\alpha+2)(\alpha+3)\kappa_{-2\alpha} - 6\eta_{-2\alpha}}{\alpha(\alpha+5)}.$$

Gauge theory (one-loop): Vacuum polarisation and Ward identities

For the convergent ILIs with α ≥ 1, the asymptotic formulae can be used to show that the gauge consistency conditions hold automatically in the limit as $\Lambda \to \infty$, as expected.

The most interesting cases to study are for α = -1, 0.

For α = 0:

$$J_0[\theta_0] - J_0[\eta_0] \sim \mathcal{M}^2 J_{-2}[\theta_0],$$
$$J_0[\kappa_0] - J_0[\eta_0] \sim 2\mathcal{M}^2 J_{-2}[\kappa_0] - \mathcal{M}^4 J_{-4}[\kappa_0].$$

After plugging in the asymptotics, we obtain logarithmic divergences that cancel. We are therefore left with the remaining finite pieces that define the following constraints

$$\gamma[\theta_0] - \gamma[\eta_0] = \frac{1}{4}, \qquad \gamma[\kappa_0] - \gamma[\eta_0] = \frac{5}{12}.$$

This gives our first glimpse at the yet to be defined relation between different η class regulators.

Gauge theory (one-loop): Vacuum polarisation and Ward identities

For α = −1 we obtain quadratic and logarithmic divergences, with the latter once again cancelling. This leaves quadratic and finite pieces:

$$\Lambda^2 C_1[\tilde{\theta}_2] + 2\mathcal{M}^2\left[\gamma[\theta_2] - \gamma[\eta_2] - \frac{1}{4}\right] \sim 0,$$
$$\Lambda^2 C_1[\tilde{\kappa}_2] + \frac{3}{2}\mathcal{M}^2\left[\gamma[\kappa_2] - \gamma[\eta_2] - \frac{5}{12}\right] \sim 0,$$

Now comes the big WOW moment:
In quite astonishing fashion, we find that in order to cancel the quadratic divergences the regulators $\tilde{\theta}_2$ and $\tilde{\kappa}_2$ are determined to be **enhanced regulators of order one.** (These are of the very same type discussed on a number theoretic level).

$$C_1[\tilde{\theta}_2] = C_1[\tilde{\kappa}_2] = 0.$$

And so we observe for the first time the connection between gauge invariance and the elimination of quadratic divergences in both divergent series and divergent loop integrals!

The remaining finite pieces then yield a set of constraints similar to what was found before:

$$\gamma[\theta_2] - \gamma[\eta_2] = \frac{1}{4}, \qquad \gamma[\kappa_2] - \gamma[\eta_2] = \frac{5}{12}.$$

Further insight into the consistency conditions arising for the finite parts, how the different η's relate, and the broader structure of η regulators, is a topic of ongoing study.

Connection to Schwinger representation

We have also found, as a preliminary result, that η regularisation can be related to Schwinger proper time.

The regularised ILIs can be written as

$$I_{-2\alpha}(\mathcal{M}^2)|_{\text{regularised}} = \frac{i}{(\alpha+1)!}\int_0^\infty \frac{d\tau}{\tau}\,\rho(\Lambda^2\tau)\tau^{2+\alpha}\int\frac{d^4k}{(2\pi)^4}e^{-\tau(k^2+\mathcal{M}^2)},$$

$$I^{\mu\nu}_{-2\alpha}(\mathcal{M}^2)|_{\text{regularised}} = \frac{i}{(\alpha+2)!}\int_0^\infty \frac{d\tau}{\tau}\,\rho(\Lambda^2\tau)\tau^{3+\alpha}\int\frac{d^4k}{(2\pi)^4}k^\mu k^\nu e^{-\tau(k^2+\mathcal{M}^2)},$$

$$I^{\mu\nu\rho\sigma}_{-2\alpha}(\mathcal{M}^2)|_{\text{regularised}} = \frac{i}{(\alpha+3)!}\int_0^\infty \frac{d\tau}{\tau}\,\rho(\Lambda^2\tau)\tau^{4+\alpha}\int\frac{d^4k}{(2\pi)^4}k^\mu k^\nu k^\rho k^\sigma e^{-\tau(k^2+\mathcal{M}^2)}$$

For the simplest choice of η we have

$$\eta_{-2\alpha}(x) \to \eta_{-2\alpha}(x,\mathcal{M}_\Lambda) = \frac{\int_0^\infty \frac{du}{u}\rho(u)[u(x^2+\mathcal{M}_\Lambda^2)]^{\alpha+2}e^{-u(x^2+M_\Lambda^2)}}{\int_0^\infty \frac{du}{u}\rho(u)[u\mathcal{M}_\Lambda^2]^{\alpha+2}e^{-uM_\Lambda^2}},$$

$$\theta_{-2\alpha}(x) \to \theta_{-2\alpha}(x,\mathcal{M}_\Lambda) = \frac{\int_0^\infty \frac{du}{u}\rho(u)[u(x^2+\mathcal{M}_\Lambda^2)]^{\alpha+3}e^{-u(x^2+M_\Lambda^2)}}{\int_0^\infty \frac{du}{u}\rho(u)[u\mathcal{M}_\Lambda^2]^{\alpha+3}e^{-uM_\Lambda^2}},$$

$$\kappa_{-2\alpha}(x) \to \kappa_{-2\alpha}(x,\mathcal{M}_\Lambda) = \frac{\int_0^\infty \frac{du}{u}\rho(u)[u(x^2+\mathcal{M}_\Lambda^2)]^{\alpha+4}e^{-u(x^2+M_\Lambda^2)}}{\int_0^\infty \frac{du}{u}\rho(u)[u\mathcal{M}_\Lambda^2]^{\alpha+4}e^{-uM_\Lambda^2}},$$

where $u = \Lambda^2\tau,\ x = |k|/\Lambda, \text{ and } \mathcal{M}_\Lambda = |\mathcal{M}|/\Lambda$

Concluding comments

- We have briefly described a new generalised, symmetry preserving regularisation prescription.
  - η-regularisation seems to capture all other common regularisation schemes, as well as a number of less common generalised prescriptions.
  - It currently resembles what one would anticipate of a ‘master regularisation’.
  - Amazingly, we have found a connection between number theory and the way to regulate divergent series (so that the sum *converges* to a unique value) and the underlying symmetries of fundamental physics!
- In addition to string theory and QFT investigations, there are a lot of interesting results in analytic number theory that we are still trying to make sense of!

Ongoing and future directions

As indicted at the outset of this talk, a lot of work is ongoing and the research programme we have defined promises many exciting future directions. Some works in progress with Tony Padilla given below:

- Second paper (forthcoming): exploring further gauge symmetry preservation and the broader structure of η regulators, including the role generalised η regulators plays with triangle anomaly.
- In collaboration with Benjamin Muntz (University of Nottingham): investigating smoothed asymptotics, the string species scale, and emergence in quantum gravity (i.e. emergence proposal from the string swampland programme).
- In collaboration with Murdock Grewar (Australian National University), exploring Fujikawa's covariant formalism to look at non-perturbatively exact renormalisation and connections with smoothed asymptotics.
- Investigating further generalities with η regularisation and the ILI programme in collaboration with Kilian Möhling (TU Dresden).

I'm also currently investigating:
-Generalisation to n-loops
-Cutkosky rules
- Unitarity, locality, and causality

-Stringy η-class regulators:
- Moduli space cutoff, moduli graphs, analytic and non-analytic structure of string amplitudes;
- String master formula;
- Gross-Mende saddle points and UV completion on the worldline.

Thanks!

Thanks for listening!