From number theory to physics: Regularising QFTs

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Based on a forthcoming series of papers in collaboration with Tony Padilla.

Motivations: UV finiteness of string amplitudes, and the longstanding hope quantum gravity will act as a universal regulator in the lower-energy limit of QFT.

Paper 3 and beyond

String theory

- String amplitudes
  - Gross-Mende
  - Handle operators
- Modular invariant regulators
- Schwinger representation and Worldline formalism

Paper 2

Quantum field theory

- η-regularisation as a general symmetry preserving method
  - Define formalism at one-loop
  - Gauge theory and master equation
- Extension to n-loop order
- Yang-Mills
- Nambu-Jona-Lasinio model

Paper 1

Analytic number theory:

- Divergent series and smooth asymptotics
  - Define infinite class of smooth cutoff functions
  - Extension from monomial series to all infinite polynomial series
- η-class regulators and analytic continuation
- Modular forms, L-series, and so on
- Resurgence theory

Divergent series

Consider the following sums of integer powers written in terms of their partial sums:

Eg. Physics tells us that the sum of the naturals should be attributed the value -1/12

$$1 + 2 + 3 + \ldots + n = \frac{1}{2}n(n+1) = \frac{1}{2}n + \frac{1}{2}n^2,$$

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{1}{6}n(n+1)(2n+1) = \frac{1}{6}n + \frac{1}{2}n^2 + \frac{1}{3}n^3,$$

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{1}{4}n^2(n+1)^2 = \frac{1}{4}n^2 + \frac{1}{2}n^3 + \frac{1}{4}n^4.$$

Classically, when analysing infinite series we might consider the definition of divergence in the Cauchy sense: 1) a series that grows in absolute value without limit or, 2) a series that is bounded but whose sequence of partial sums does not approximate any specific value.

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n.$$

Divergent series

Tao, 2011

If one wants to therefore make sense of divergent series, the task is in a certain sense to develop a theory that matches the partial sum behaviour of finite sums, but allows the result to be generalised to infinity.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \ldots .$$

More generally, after analytic continuation:

$$\zeta(-s) = -\frac{B_{s+1}}{s+1}$$

The majority of issues with divergent series comes with the transition from partial sums to infinity, as notably exposed by Ramanujan. Ramanujan, like Euler, was fascinated with formal manipulation of infinite series.

If one formally applies values s = 1,2,… one will obtain:

$$\sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + \ldots = -1/2$$

$$\sum_{n=1}^{\infty} n = 1 + 2 + 3 + \ldots = -1/12,$$

$$\sum_{n=1}^{\infty} n^2 = 1 + 4 + 9 = 0.$$

Much has been said of these rather bizarre, if not altogether absurd, formulae. The do not appear coherent or reasonable. Observe, for instance, as written these formulae are characterised by the fact that positive summands appear to equate to some negative or zero value.

Tao's method of smooth summation

Tao, 2011

Instead we consider smooth sums of the form $$\sum_{n=1}^{\infty} \eta\left(\frac{n}{N}\right) a_n$$

in which the notion of convergence is now defined as $$\lim_{N\to\infty} \sum_{n=1}^{\infty} a_n \eta\left(\frac{n}{N}\right) = s.$$

We define a smooth function η(x) : R+ → R that is a bounded function of compact support, which is taken to be the interval [0,1]. This means that x → ∞ η(x) = 0 and η(0) = 1.

This leads to the generalised Euler-Maclaurin formula

$$\int_0^N f(x)\eta(x)\,dx = \frac{1}{2}f(0) + \sum_{k=1}^{N} f(k)\eta(k) + \sum_{i=2}^{s+1} \frac{B_i}{i!} f^{(i-1)}(0) + O(N \,||\, f^{(s+2)} \,||_\infty),$$

where $|| \, f^{(s+2)} \, ||_\infty = \sup_{x\in\mathbb{R}} | \, f^{(s+2)(x)} \, |.$

Relation to the zeta function

From the generalised Euler-Maclaurin formula, smooth summation can be related to the Riemann zeta function:

Tao, 2011

$$\zeta(s) = \lim_{N\to\infty} \sum_{n=1}^{\infty} \frac{1}{n^s}\eta\left(\frac{n}{N}\right) - N^{1-s}\mathcal{M}_\eta(-s),$$ or, more conveniently,

$$\sum_{n=0}^{\infty} n^s \ \eta\left(\frac{n}{N}\right) = C_{\eta,s}N^{s+1} + \zeta(-s) + O\left(\frac{1}{N}\right),$$ where $$C_{\eta,s}N^{s+1} = N^{s+1}\int_0^\infty \ dx \ x^s \ \eta(x)$$

From the fact that the overall divergent sum can be decomposed into a finite part and an infinite part, it is clear that the finite piece doesn't depend on the choice of regulator.

This is true if η(x) is a sufficiently smooth function.

The divergent pieces of the smooth partial sums scale like $C_{\eta,s}N^{(s+1)}$ which is scheme dependent. So for the correct choice of cutoff the integral can be killed completely, thus exposing the unique finite value of the divergent series as perhaps the most natural value.

Example: Sums of powers

So the task to regularise any divergent series essentially reduces to the appropriate choice of regulator that kills the integral

$$C_{\eta,s} := \int_0^\infty x^s \eta(x)\ dx$$

As an example, for the sums of integer powers it is easily found

$$s = 0, \quad 1 + 1 + 1 + 1 + \ldots + 1 \overset{\eta}{=} -\frac{1}{2} + C_{\eta,0} + O\left(\frac{1}{N}\right),$$

$$s = 1, \quad 1 + 2 + 3 + 4 + \ldots + N \overset{\eta}{=} -\frac{1}{12} + C_{\eta,1} N^2 + O\left(\frac{1}{N}\right),$$

$$s = 2, \quad 1 + 2^2 + 3^2 + 4^2 + \ldots + N^2 \overset{\eta}{=} 0 + C_{\eta,2} N^3 + O\left(\frac{1}{N}\right).$$

Extending Tao's method

We show that many of the key ideas of Tao's method can be extended. Two important extensions to Tao's method for number theoretic sums:

1) We observe that the regulator cutoff may be interpreted as entering via a modification of the integration measure. From a physics point of view, this interpretation shares the philosophy of dimensional regularisation and is mathematically similar to the concept of mollification.

$$\int_0^\infty dx \to \int_0^\infty dx\ \eta(x)$$

2) We show (with proof) that the smooth cutoff function can be extended to the much more general class of Schwartz functions.

$$\mathcal{S}(\mathbb{R}^n, \mathbb{C}) := \left\{ f \in C^\infty(\mathbb{R}^n, \mathbb{C}) \mid \forall \alpha, \beta \in \mathbb{N}^n, || f ||_{\alpha,\beta} < \infty \right\}$$ where

$$|| f ||_{\alpha,\beta} := \sup_{x \in \mathbb{R}^n} | x^\alpha (D^\beta f)(x) |$$

Then $\eta : \mathbb{R} \mapsto \mathbb{R}$ is defined to be a Schwartz function if for every r ≥ 0 the rth derivative exists and goes rapidly to zero.

Furthermore, the class of smooth cutoff functions η can be any C^∞ function so long that we can also ensure η(0)=1.

Generalising to QFTs: Naïve example

Example at one-loop

$$\mathcal{M} = \frac{(i\lambda)^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2} \frac{i}{(k+p)^2 - m^2}$$

After Feynman parameterisation and Wick rotating to Euclidean space

$$I = \frac{(-i\lambda)^2}{2} \int_0^1 dx \int d^4k \frac{-i}{(k^2 + M^2)^2}.$$

We can now follow our regularisation procedure by including a smooth cutoff

$$I^R = \frac{(-i\lambda)^2}{2} \int_0^1 dx \int d^4k \frac{-i}{(k^2 + M^2)^2} \eta(\epsilon k)$$

Using change of variables and some minor computation we have two integrals

$$I^R = \frac{-i^3\lambda^2\epsilon^{-2}}{2} \int_0^1 dx \int dq\ q\ \eta(q) - M^2 \int dq\ \frac{1}{q}\eta(q) \sim -M^2 \log(q) + \text{higher order corrections.}$$

The naïve observation is that we can regularise any divergent Feynman integral at one-loop.

- It is generally easy to regularise divergent integrals. The hard part is to do it consistently and for all QFTs.
- Any regularisation prescription worth its salt should satisfy locality, causality, and the Ward identity.
- So the question is, how do we formalise these naïve observations?

Irreducible loop integrals (one-fold ILIs) at one-loop

In order to define a consistent and useful regularisation prescription, we utilise the concept of irreducible loop integrals (ILIs) as first introduced in [0209021 - Wu].

In general, the set of ILIs can be written as follows:

$$I_{-2\alpha} = \int d^4k \frac{1}{(k^2 - M^2)^{2+\alpha}}$$

$$I_{-2\alpha\mu\nu} = \int d^4k \frac{k_\mu k_\nu}{(k^2 - M^2)^{3+\alpha}}$$

$$I_{-2\alpha\mu\nu\rho\sigma} = \int d^4k \frac{k_\mu k_\nu k_\rho k_\sigma}{(k^2 - M^2)^{4+\alpha}}$$

- The mass factor M is a function of Feynman parameters, external momenta, and corresponding mass scales.
- The number (-2α) in the subscript labels the power counting dimension (of energy-momentum).
  - Here α = -1,0,1,2,… and, for the case where α = 0 and α = -1, one obtains the corresponding logarithmically and quadratically divergence integrals at one-loop, respectively.

- All one-loop Feynman integrals can be reduced to their respective ILIs.
- The concept of ILIs can be generalised to arbitrary loop order.

Regularised ILIs at one-loop

To regularise any divergent Feynman integral at one-loop, we must first apply Feynman parametrisation (sometimes repeatedly) to the amplitude as a whole, then reduce the integral to the appropriate ILIs.

Given that all one-loop integrals can be expressed in terms of the one-fold ILIs by way of the Feynman parameter method, we see that the divergences are thus completely characterized by these one-fold ILIs.

- Note: all divergent Feynman integrals are evaluated in Euclidean space. So once we have the integral represented as its appropriate ILI, we perform a Wick rotation as usual and then proceed to compute the integral.

For example, we have the following master set of regularised integrals as related to the set of ILIs at one-loop:

$$I_2^R = -iJ_2^R[\eta_2] = -i\int d^4k\ \frac{1}{k^2+M^2}\eta_2\left(\frac{|k|}{\Lambda}\right)$$

$$I_0^R = iJ_0^R[\eta_0] = i\int d^4k\ \frac{1}{(k^2+M^2)^2}\eta_0\left(\frac{|k|}{\Lambda}\right)$$

$$I_{2\mu\nu}^R = -iJ_{2\mu\nu}^R[\tilde{\eta}_2] = -i\int d^4k\ \frac{k^2}{(k^2+M^2)^2}\tilde{\eta}_2\left(\frac{|k|}{\Lambda}\right)$$

Here we have made the simplest choice for η(x) such that x = k/Λ. More general choices are possible.

Generalising to QFTs: Introducing η-regularisation

At one-loop, the general procedure to regulate any divergent Feynman integral entails:

1. use Feynman parametrisation and shift the integration variable to reduce the integral to its corresponding ILI;
2. Wick rotate from four-dimensional Minkowski spacetime to four-dimensional Euclidean space;
3. evaluate the perturbative Feynman integrals in terms of their corresponding ILIs by replacing the loop integration measure

$$\int d^4k \to \int d^4k \; \eta(x)$$

4. further decompose the integral if necessary, and use change of variables.

Gauge theory: Ward identity (one-loop)

Specialising to the case of QED we can write the vacuum polarisation function

$$\Pi_{\mu\nu}(k) = k_\mu k_\nu \Pi(k^2) - g_{\mu\nu} k^2 \Pi(k^2)$$

which we can write in terms of ILIs

$$= \chi \int_0^1 dx \left[2I_{2\mu\nu}(m) - I_2(m) g_{\mu\nu} + 2x(1-x)(p^2 g_{\mu\nu} - p_\mu p_\nu) I_0(m) \right]$$

The terms proportional to p satisfy the Ward identity, and so we are led to the condition

$$2I_{2\mu\nu}(m) - I_2(m) g_{\mu\nu} = 0$$

One interesting result is that we derive a master equation written in η-language

$$\lim_{\Lambda \to 0} \{ M^2 \partial_{M^2} J_2^R[\eta] - J_2^2[\bar{\eta}] \} = 0$$

Any regularisation prescription that is symmetry preserving must satisfy this equation. For example, there is a choice of η that captures dimensional regularisation. As expected, we find dim reg satisfies this equation.

In satisfying this equation for general η, we find a term that must be killed. This term happens to be precisely an enhanced regulator of order one (from previous number theory discussion)! And so it is killed and everything works out!

Generalising to 2-loops and higher

Problem: the main issue with the ILI formalism at higher loop order is the treatment of overlapping divergences.

Theorem [Wu, 2002; Huang et al., 2013; Bai and Wu, 2017;]: a divergent Feynman integral at 2-loop order and higher, the structures of UV contributions can be extracted as

$$I = F_{FP} \otimes I_{UVDP} \otimes I_{ILI}$$

At 1-loop since there are no sub-divergences so we simply have $I = I_{\mathrm{FP}} \otimes I_{\mathrm{ILI}}$

Crucially, at multiloop level sub-contributions are completely captured by the UVDP integral. A key observation that makes the ILI formalism work at n-loops is the fact that the ILI and UVDP integrals actually comprise αβγ diagrams of ‘t Hooft and Veltman [‘t Hooft and Veltman 72, Ashmore 72, Bollini and Giambiaggi 72]:

$$I_{UVDP} \otimes I_{ILI} = I_{\alpha\beta\gamma}$$

Theorem [Huang et al., 2013]: There is a one-to-one correspondence between sub-contributions in αβγ diagrams and those in UVDP integrals.

We leverage these arguments, and use the ILI programme to extend the generalised η-regularisation to arbitrarily loop order.

Generalising to 2-loops and higher

The easiest way to see this is to review 2-loop calculation, because the extension from 2-loops to arbitrary loop order is straightforward. [Wu, 2002]

Using Feynman parameterisation higher loop integrals can in general be expressed in the form of the following general overlapping integrals known as α, β, γ diagrams

$$I^{(2)}_{\alpha\beta\gamma} = \int d^4k_1 \int d^4k_2 \frac{1}{(k_1^2+M_1^2)^\alpha (k_2^2+M_2^2)^\beta \left((k_1-k_2+p)^2+M_{12}^2\right)^\gamma}$$

Using the formal for Feynman parameters

$$\frac{1}{a_1^{\alpha_1} a_2^{\alpha_2} \dots a_n^{\alpha_n}} = \frac{\Gamma(\alpha_1 \dots \alpha_n)}{\Gamma(\alpha_1)\Gamma(\alpha_n)} \int_0^1 dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{n-2}} dx_{n-1} \frac{(1-x_1)^{\alpha_1-1}(x_1-x_2)^{\alpha_2-1}\dots x_{n-1}^{\alpha_n-1}}{[a_1(1-x_1)+a_2(x_1-x_2)+\dots+a_n x_{n-1}]^{\alpha_1+\dots+\alpha_n}}$$

We can rewrite the α β γ integral

$$I^{(2)}_{\alpha\beta\gamma} = \frac{\Gamma(\alpha+\gamma)}{\Gamma(\alpha)\Gamma(\gamma)} \int_0^1 dx \int d^4k_1 \int d^4k_2 \frac{(1-x)^{\alpha-1}x^{\gamma-1}}{[k_1^2+x(1-x)(k_2-p)^2+M_x^2]^{\alpha+\gamma}(k_2^2+M_2^2)^\beta}$$

Generalising to 2-loops and higher

To get this integral in sufficient ILI form, we reparameterise the divergence and transform it to u-space (i.e., UV divergence preserving parameterisation. In the present case, we can use the formula

$$\frac{1}{a^\alpha b^\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 du \, \frac{u^{\beta-1}}{[a+bu]^{\alpha+\beta}}$$

[Wu, 2002]

After some work, the 2-loop scalar-like integral takes the form

$$I^{(2)}_{\alpha\beta\gamma} = \frac{\Gamma(\alpha+\beta+\gamma-2)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int_0^1 dx \; x^{\gamma-1}(1-x)^{\alpha-1}(q_0^2)^{\alpha+\gamma-2} \sum_{i=0}^{\beta-1} c_i^{\beta-1}(x(1-x)q_0^2)^i$$

$$\int \frac{d^4\hat{k}_1}{\hat{k}_1^2} \frac{1}{(\hat{k}_1^2 + x(1-x)q_0^2)^{\alpha+\gamma-1+i}} \int d^4 k_2 \; \frac{1}{(k_2^2 + M_2^2 + \mu^2_{\hat{k}_1^2})^{\alpha+\beta+\gamma-2}},$$

in which it is noticed that the final k_2 integral is a one-fold ILI for the two-loop graphs. It follows now that we can regularise, similarly as in the 1-loop case.

To give a quick example of what the regularised integral looks like in this case, upon introducing a smooth cutoff and using change of variables we have

$$I^{(2)R}_{\alpha\beta\gamma} = \frac{\Gamma(\alpha+\beta+\gamma-2)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int_0^1 dx\ x^{\gamma-1}(1-x)^{\alpha-1}(q_0^2)^{\alpha+\gamma-2} \sum_{i=0}^{\beta-1} c_i^{\beta-1}(x(1-x)q_0^2)^i$$

$$\frac{1}{\hat{\Lambda}_1^3} \int \frac{d^4\hat{x}_1}{\hat{x}_1^2} \frac{1}{(\hat{x}_1^2 + x(1-x)q_0^2)^{\alpha+\gamma-1+i}} \eta(\hat{x_1})$$

$$\frac{1}{\Lambda_2} \int d^4k_2\ \frac{1}{(x_2^2 + M_2^2/\Lambda_2 + \mu^2_{\hat{k}_1^2})^{\alpha+\beta+\gamma-2}} \eta(x_2)$$

Note that depending on the value of α,β,γ the integrals will be logarithmically or quadratically divergent.

n-fold ILIs

Much more generally, we can define n-fold ILIs. For example, one form of the scalar-like n-fold ILI is given as

$$I^{(n)}_{\alpha_i \alpha_{ij}} = \int d^4 k_n \, \frac{1}{(k_n^2 + M_n^2)^{\alpha_n}} \prod_{j>i} \prod_{i=1}^{n-1} \int d^4 k_i \, \frac{1}{(k_i^2 + M_i^2)^{\alpha_i}} \frac{1}{[(k_i - k_j + p_{ij})^2 + M_{ij}^2]^{\alpha_{ij}}}$$

Wu, 2003

We can similarly write tensor-like n-fold ILIs. Upon use of UVDP parameterisation we have for both cases

$$I^{(n)}_{\Delta} = \prod_{i=1}^{n-1} \int_0^\infty du_i \, \frac{F_{is}(x_{lm})}{(u_i + \rho_i)^{\Delta_{is}}} I^{(1)}_{\Delta_n}(\mu_n^2)$$

$$I^{(n)}_{\Delta\mu\nu} = \prod_{i=1}^{n-1} \int_0^\infty du_i \, \frac{F_{is}(x_{lm})}{(u_i + \rho_i)^{\Delta_{is}}} I^{(1)}_{\Delta_n \mu\nu}(\mu_n^2),$$

where

$$I^{(1)}_{\Delta_n}(\mu_n^2) = \int d^4 k_n \, \frac{1}{(k_n^2 + M^2 + \mu_n^2)^{\Delta_n}}$$

$$I^{(1)}_{\Delta_n \mu\nu}(\mu_n^2) = \int d^4 k_n \, \frac{k_{n\mu} k_{n\nu}}{(k_n^2 + M^2 + \mu_n^2)^{\Delta_n + 1}}$$

Then the regularisation proceeds similarly to what was discussed before.

Stringy regulators

From the view of studying string amplitudes, an early motivation was the observation that natural choices of η are typically multi-term exponentially decreasing functions. These tend to look quite stringy.

Indeed, from the view of string amplitudes, this sort of general exponential damping is what makes the scattering amplitudes soft in the UV. There are a number of insightful ways to probe this fact.

- For instance, string field theory propagators can be found to be dressed with an exponential decreasing term.
- Another example [Abel and Lewis, 2020]: in the Gross-Mende regime these exponential damped amplitudes can be seen quite explicitly. One can see that this exponential damping, which comes about as the exponential correction in the Green function (lowest mode in the tower of states) is associated with a saddle point in moduli space (a signature of stringy behaviour).
  - We start to see in this analysis that we can basically 'pick off' stringy η's. The idea is then to map them to QFT via the worldline formalism.
- Another interest bit of work involves the handle operator approach to string amplitudes [Skliros and Lust, 2019].
  - Highly technical; in short, it is possible to extract the Feynman propagator and ongoing work is investigating the way in which stringy η's can be preserved as an artefact of UV finiteness on the level low-energy QFT.
- We have also found ways to define modular invariant η-class regulators!

These are just a few examples of ongoing work that completes η-regularisation as a possible universal regulator from quantum gravity!

Concluding comments

- We have briefly described a new generalised, symmetry preserving regularisation prescription.
  - η-regularisation captures all other common regularisation schemes, as well as a number of less common generalised prescriptions.
  - It currently resembles what one would anticipate of a ‘master regularisation’.

- In addition to string theory and QFT, there are a lot of interesting results in analytic number theory that we are still trying to make sense of!
  - Work is ongoing to better understand numerous sum and integral identities.
  - Examples: further study of η-regularisation and its relation to analytic continuation.
    - Modular forms
    - Resurgence theory