From number theory to physics: Regularising QFTs

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The Nottingham Centre of Gravity, 15 December 2023

Based on a forthcoming series of papers in collaboration with Tony Padilla.

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Paper 4 and beyond

Motivations: UV finiteness of string amplitudes, and the longstanding hope quantum gravity will act as a universal regulator in the lower-energy limit of QFT.

Paper 3 — String theory

- String amplitudes
  - Gross-Mende
  - Handle operators
- Modular invariant regulators
- Schwinger representation and Worldline formalism

Paper 2 — Quantum field theory

- η-regularisation as a general symmetry preserving method
  - Define formalism at one-loop
  - Gauge theory and master equation
- Extension to n-loop order
- Yang-Mills
- Nambu-Jona-Lasinio model

Paper 1 — Analytic number theory:

- Divergent series and smooth asymptotics
  - Define infinite class of smooth cutoff functions
  - Extension from monomial series to all infinite polynomial series
- η-class regulators and analytic continuation
- Modular forms, L-series, and so on
- Resurgence theory

Sums of integer powers

Consider the following sums of integer powers written in terms of their partial sums:

Examples in physics tells us that the sum of the naturals should be attributed the value -1/12.

$$1+2+3+\ldots+n=\frac{1}{2}n(n+1)=\frac{1}{2}n+\frac{1}{2}n^2,$$

$$1^2+2^2+3^2+\ldots+n^2=\frac{1}{6}n(n+1)(2n+1)=\frac{1}{6}n+\frac{1}{2}n^2+\frac{1}{3}n^3,$$

$$1^3+2^3+3^3+\ldots+n^3=\frac{1}{4}n^2(n+1)^2=\frac{1}{4}n^2+\frac{1}{2}n^3+\frac{1}{4}n^4.$$

Classically, when analysing infinite series we might consider the definition of divergence in the Cauchy sense: 1) a series that grows in absolute value without limit or, 2) a series that is bounded but whose sequence of partial sums does not approximate any specific value.

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n.$$

Divergent series

Tao, 2011

Compare the partial sums with the values given by the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \ldots.$$

More generally, after analytic continuation: $\zeta(-s) = -\frac{B_{s+1}}{s+1}$

Formally applying values s = 1,2,… one will obtain:

$$\sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + \ldots = -1/2$$

$$\sum_{n=1}^{\infty} n = 1 + 2 + 3 + \ldots = -1/12,$$

$$\sum_{n=1}^{\infty} n^2 = 1 + 4 + 9 = 0.$$

$$\sum_{n=1}^{\infty} n^s = 1^s + 2^s + 3^s + \cdots = -\frac{B_{s+1}}{s+1}$$

Much has been said of these rather bizarre, if not altogether absurd, formulae. They were of course also made famous by Ramanujan.

- They do not appear coherent or reasonable because, as written, these formulae are characterised by positive summands on the left-hand side appearing to equate to some negative or zero value.
- Comparing with the previous partial sums, one can try to inspect the partial sums of these divergent series. But there is no obvious relationship with these constant values.

Compare divergent series and partial sums

Tao, 2011

Define $S_s(N) = \sum_{k=1}^{N} k^s$ then the previous partial sums can be expressed as special cases of Faulhaber's formula

$$S_s(N) = \frac{1}{s+1}\sum_{k=0}^{s}\binom{s+1}{k} B_k N^{s+1-k}$$

where B_k denotes the Bernoulli numbers.

$$= \frac{1}{s+1}N^{s+1} + \frac{1}{2}N^s + \frac{s}{12}N^{s-1} + \ldots + B_s N$$

In the limit N -> ∞ Faulhaber's formula breaks down. Quite simply, this is because in the Cauchy sense an infinite series is divergent.

Furthermore, in a lovely bit of analysis comparing the partial sums behaviour with the divergent series, Tao shows:
If N is considered a real number, then this sum has jump discontinuities (of the first kind) at each positive integer value of N.

- These discontinuities produce various artefacts when trying to expand $\sum_{k=1}^{N} k^s = \sum_{k \leq N} k^s$
- In the traditional partial sums, these artefacts arise due to discretisation as a result of the abrupt truncation of the sum at some N.

Tao's method of smooth summation

Tao, 2011

Instead we consider smooth sums of the form $\sum_{n=1}^{\infty} \eta\left(\frac{n}{N}\right) a_n$

in which the notion of convergence is now defined as $\lim_{N\to\infty} \sum_{n=1}^{\infty} a_n \eta\left(\frac{n}{N}\right) = s.$

We define a smooth function η(x) : R+ → R that is a bounded function of compact support, which is taken to be the interval [0,1]. This means that x → ∞ η(x) = 0 and η(0) = 1.

This leads to the generalised Euler-Maclaurin formula

$$\int_0^N f(x)\eta(x)\, dx = \frac{1}{2} f(0) + \sum_{k=1}^{N} f(k)\eta(k) + \sum_{i=2}^{s+1} \frac{B_i}{i!} f^{(i-1)}(0) + O(N \mid\mid f^{(s+2)} \mid\mid_\infty),$$

where $\mid\mid f^{(s+2)} \mid\mid_\infty = \sup_{x\in\mathbb{R}} \mid f^{(s+2)(x)} \mid.$

Relation to the zeta function

From the generalised Euler-Maclaurin formula, smooth summation can be related to the Riemann zeta function:

Tao, 2011

$$\zeta(s) = \lim_{N\to\infty} \sum_{n=1}^{\infty} \frac{1}{n^s}\eta\left(\frac{n}{N}\right) - N^{1-s}\mathcal{M}_\eta(-s),$$ or, more conveniently,

$$\sum_{n=0}^{\infty} n^s\ \eta\left(\frac{n}{N}\right) = C_{\eta,s}N^{s+1} + \zeta(-s) + O\left(\frac{1}{N}\right),$$ where $$C_{\eta,s}N^{s+1} = N^{s+1}\int_0^\infty\ dx\ x^s\ \eta(x)$$

The overall divergent sum can be decomposed into a finite part and an infinite part. The finite piece doesn't depend on the choice of regulator.

This is true if η(x) is a sufficiently smooth function.

The divergent pieces of the smooth sums scale like $C_{\eta,s}N^{(s+1)}$ which is scheme dependent. So for the correct choice of cutoff the integral can be killed completely, thus exposing the unique finite value of the divergent series as perhaps the most natural value.

Example: Sums of powers

So the task to regularise any divergent series essentially reduces to the appropriate choice of regulator that kills the integral

$$C_{\eta,s} := \int_0^\infty x^s \eta(x)\ dx$$

As an example, for the sums of integer powers it is easily found

$$s = 0, \quad 1 + 1 + 1 + 1 + \ldots + 1 \overset{\eta}{=} -\frac{1}{2} + C_{\eta,0} + O\left(\frac{1}{N}\right),$$

$$s = 1, \quad 1 + 2 + 3 + 4 + \ldots + N \overset{\eta}{=} -\frac{1}{12} + C_{\eta,1}N^2 + O\left(\frac{1}{N}\right),$$

$$s = 2, \quad 1 + 2^2 + 3^2 + 4^2 + \ldots + N^2 \overset{\eta}{=} 0 + C_{\eta,2}N^3 + O\left(\frac{1}{N}\right).$$

Extending Tao's method

We show that many of the key ideas of Tao's method can be extended. Two important extensions to Tao's method for number theoretic sums:

1) We observe that the regulator cutoff may be interpreted as entering via a modification of the integration measure. From a physics point of view, this interpretation shares the philosophy of dimensional regularisation and is mathematically similar to the concept of mollification.

$$\int_0^\infty dx \to \int_0^\infty dx\ \eta(x)$$

2) We show (with proof) that the smooth cutoff function can be extended to the much more general class of Schwartz functions.

$$\mathcal{S}(\mathbb{R}^n, \mathbb{C}) := \left\{ f \in C^\infty(\mathbb{R}^n, \mathbb{C}) \mid \forall \alpha, \beta \in \mathbb{N}^n, ||\ f\ ||_{\alpha,\beta} < \infty \right\}$$ where

$$||\ f\ ||_{\alpha,\beta} := \sup_{x \in \mathbb{R}^n} |\ x^\alpha (D^\beta f)(x)\ |$$

Then $\eta : \mathbb{R} \mapsto \mathbb{R}$ is defined to be a Schwartz function if for every r ≥ 0 the rth derivative exists and goes rapidly to zero.

Furthermore, the class of smooth cutoff functions η can be any C^∞ function so long that we can also ensure η(0)=1.

Another extension: Polynomial series

Consider the infinite sum $\sum_{k=1}^{\infty} f(k)$ where $f(k) = \sum_{s=0}^{z} c_s k^s$ is a polynomial of degree z.

From the generalised Euler-Maclaurin formula in which we define $g_N(x) = f(x)\eta\left(\frac{x}{N}\right)$ we show with proof

$$\sum_{k=1}^{\infty} f(k)\eta\left(\frac{k}{N}\right) = \sum_{s=0}^{z} c_s[C_{s,\eta}N^{s+1} + \zeta(-s)] + O\left(\frac{1}{N}\right)$$

Due to extending $\eta(x)$ to be Schwartz, it can be shown crucially that the integral in the remainder term of the Euler-Maclaurin formula is bounded and goes like O(1/N).

Importantly, we see $C_{s,\eta} = \int_0^{\infty} dx\ x^s \eta(x)$ is the Mellin transform of the smooth regulator function.

Observation: We find in general that power law divergences are regulator dependent and weighted by the corresponding Mellin transform.

- This is a feature reminiscent of QFT.

- Interestingly, the regulator dependence in the above raises the possibility that there are families of enhanced regulators for which the divergences vanish altogether!

Enhanced regulators

Definition: an enhanced regulator is one for which the Mellin transform $C_{s,\eta} = \int_0^\infty dx\, x^s \eta(x)$ vanishes for integer values of s ≥ 0.

An extremely elegant example of an enhanced regulator of order z is given by

$$\eta_s(x) = e^{-x\cot\left(\frac{\frac{\pi}{2}-\theta}{s+1}\right)} \frac{\cos(x+\theta)}{\cos\theta}$$

where $0 < \theta < \frac{\pi}{2}$ and s is any natural number.

For θ = 0 and s = 1 we recover the astonishingly beautiful enhanced regulator of order one $\eta_1(x) = e^{-x}\cos x$

from which it can be inferred $\lim_{N\to\infty} \sum_{n=1}^{\infty} n e^{-\frac{n}{N}} \cos\left(\frac{n}{N}\right) = -\frac{1}{12}$ for the sum of natural numbers.

Note: We also define super-enhanced regulators for polynomial series, and for both monomial and polynomial cases we have defined an algorithm for finding enhanced and super-regulators of any given order given for any Schwartz function.

Generalising to QFTs: Naïve example

Example at one-loop

$$\mathcal{M} = \frac{(i\lambda)^2}{2}\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2 - m^2}\frac{i}{(k+p)^2 - m^2}$$

After Feynman parameterisation and Wick rotating to Euclidean space

$$I = \frac{(-i\lambda)^2}{2}\int_0^1 dx \int d^4k \frac{-i}{(k^2+M^2)^2}.$$

We can now follow our regularisation procedure by including a smooth cutoff

$$I^R = \frac{(-i\lambda)^2}{2}\int_0^1 dx \int d^4k \frac{-i}{(k^2+M^2)^2}\eta(\epsilon k)$$

Using change of variables and some minor computation we have two integrals

$$I^R = \frac{-i^3\lambda^2\epsilon^{-2}}{2}\int_0^1 dx \int dq\; q\; \eta(q) - M^2 \int dq\; \frac{1}{q}\eta(q) \sim -M^2\log(q) + \text{higher order corrections.}$$

The naïve observation is that we can regularise any divergent Feynman integral at one-loop.

- It is generally easy to regularise divergent integrals. The hard part is to do it consistently and for all QFTs.
- Any regularisation prescription worth its salt should satisfy locality, causality, and the Ward identity.
- So the question is, how do we formalise these naïve observations?

Irreducible loop integrals (one-fold ILIs) at one-loop

In order to define a consistent and useful regularisation prescription, we utilise the concept of irreducible loop integrals (ILIs) as first introduced in [Wu, 2002].

In general, the set of ILIs can be written as the following master integrals:

$$I_{-2\alpha} = \int d^4k \frac{1}{(k^2 - M^2)^{2+\alpha}}$$

$$I_{-2\alpha\mu\nu} = \int d^4k \frac{k_\mu k_\nu}{(k^2 - M^2)^{3+\alpha}}$$

$$I_{-2\alpha\mu\nu\rho\sigma} = \int d^4k \frac{k_\mu k_\nu k_\rho k_\sigma}{(k^2 - M^2)^{4+\alpha}}$$

- The mass factor M is a function of Feynman parameters, external momenta, and corresponding mass scales.
- The number (-2α) in the subscript labels the power counting dimension (of energy-momentum).
  - Here α = -1,0,1,2,... and, for the case where α = 0 and α = -1, one obtains the corresponding logarithmically and quadratically divergence integrals at one-loop, respectively.
- All one-loop Feynman integrals can be reduced to their respective ILIs.
- The concept of ILIs can be generalised to arbitrary loop order.

Regularised ILIs at one-loop

To regularise any divergent Feynman integral at one-loop, we must first apply Feynman parametrisation (sometimes repeatedly) to the amplitude as a whole, then reduce the integral to the appropriate ILIs.

Given that all one-loop integrals can be expressed in terms of the one-fold ILIs by way of the Feynman parameter method, we see that the divergences are thus completely characterized by these one-fold ILIs.

- Note: all divergent Feynman integrals are evaluated in Euclidean space. So once we have the integral represented as its appropriate ILI, we perform a Wick rotation as usual and then proceed to compute the integral.

For example, we have the following master set of regularised integrals as related to the set of ILIs at one-loop:

$$I_2^R = -iJ_2^R[\eta_2] = -i\int d^4k\, \frac{1}{k^2+M^2}\eta_2\left(\frac{|k|}{\Lambda}\right)$$

$$I_0^R = iJ_0^R[\eta_0] = i\int d^4k\, \frac{1}{(k^2+M^2)^2}\eta_0\left(\frac{|k|}{\Lambda}\right)$$

$$I_{2\mu\nu}^R = -iJ_{2\mu\nu}^R[\tilde{\eta}_2] = -i\int d^4k\, \frac{k^2}{(k^2+M^2)^2}\tilde{\eta}_2\left(\frac{|k|}{\Lambda}\right)$$

Here we have made the simplest choice for η(x) such that x = k/Λ. More general choices are possible.

- $|k|$ the norm of the Euclidean four-momentum and Λ is the cut-off scale.

Generalising to QFTs: Introducing η-regularisation

At one-loop, the general procedure to regulate any divergent Feynman integral entails:

1. use Feynman parametrisation and shift the integration variable to reduce the integral to its corresponding ILI;
2. Wick rotate from four-dimensional Minkowski spacetime to four-dimensional Euclidean space;
3. evaluate the perturbative Feynman integrals in terms of their corresponding ILIs by replacing the loop integration measure

$$\int d^4k \to \int d^4k \; \eta(x)$$

4. further decompose the integral if necessary, and use change of variables.

Gauge theory: Ward identity (one-loop)

Specialising to the case of QED we can write the vacuum polarisation function

$$\Pi_{\mu\nu}(k) = k_\mu k_\nu \Pi(k^2) - g_{\mu\nu} k^2 \Pi(k^2)$$

which we can then write in terms of ILIs

$$= \chi \int_0^1 dx \left[2I_{2\mu\nu}(m) - I_2(m) g_{\mu\nu} + 2x(1-x)(p^2 g_{\mu\nu} - p_\mu p_\nu) I_0(m) \right]$$

Terms proportional to p satisfy the Ward identity, so we are led to the condition

$$2I^R_{2\mu\nu}(m) - I^R_2(m) g_{\mu\nu} = 0$$

For general η we derive the following master equation

$$\lim_{\Lambda\to\infty} \{ M^2 \partial_{M^2} J^R_2[\eta] - J^R_2[\bar{\eta}] \} = 0$$

Any regularisation prescription that is symmetry preserving must satisfy this equation.

- Interestingly, the class of η-regulators is so general that it seems to capture all common regularisation schemes.
  - E.g. there is a choice of η that captures dimensional regularisation. As expected, we find dim reg satisfies this equation.

In satisfying this equation for general η, we find a term that must be killed. This term happens to be precisely an enhanced regulator of order one, and so it can be killed. Amazingly, then, the enhanced regulators defined on a number theory level are found to be required to preserve gauge invariance!

Generalising to 2-loops and higher

Problem: the main issue with the ILI formalism at higher loop order is the treatment of overlapping divergences.

Wu, 2002

Theorem [Wu, 2002; Huang et al., 2013; Bai and Wu, 2017;]: a divergent Feynman integral at 2-loop order and higher, the structures of UV contributions can be extracted as

$$I = F_{FP} \otimes I_{UVDP} \otimes I_{ILI}$$

At 1-loop since there are no sub-divergences so we simply have $I = I_{\mathrm{FP}} \otimes I_{\mathrm{ILI}}$

Crucially, at multiloop level sub-contributions are completely captured by the UVDP integral. A key observation that makes the ILI formalism work at n-loops is the fact that the ILI and UVDP integrals actually comprise αβγ diagrams of ‘t Hooft and Veltman [‘t Hooft and Veltman 72, Ashmore 72, Bollini and Giambiaggi 72]:

$$I_{UVDP} \otimes I_{ILI} = I_{\alpha\beta\gamma}$$

Theorem [Huang et al., 2013]: There is a one-to-one correspondence between sub-contributions in αβγ diagrams and those in UVDP integrals.

We leverage these arguments, and use the ILI programme to extend the generalised η-regularisation to arbitrarily loop order.

Generalising to 2-loops and higher

The easiest way to see this is to review 2-loop calculation, because the extension from 2-loops to arbitrary loop order is straightforward.

Wu, 2002

Using Feynman parameterisation higher loop integrals can in general be expressed in the form of the following general overlapping integrals known as α, β, γ diagrams

$$I^{(2)}_{\alpha\beta\gamma} = \int d^4k_1 \int d^4k_2 \frac{1}{(k_1^2+M_1^2)^\alpha (k_2^2+M_2^2)^\beta \left((k_1-k_2+p)^2+M_{12}^2\right)^\gamma}$$

Using the formal for Feynman parameters

$$\frac{1}{a_1^{\alpha_1} a_2^{\alpha_2} \dots a_n^{\alpha_n}} = \frac{\Gamma(\alpha_1 \dots \alpha_n)}{\Gamma(\alpha_1)\Gamma(\alpha_n)} \int_0^1 dx_1 \int_0^{x_1} dx_2 \dots \int_0^{x_{n-2}} dx_{n-1} \frac{(1-x_1)^{\alpha_1-1}(x_1-x_2)^{\alpha_2-1}\dots x_{n-1}^{\alpha_n-1}}{[a_1(1-x_1)+a_2(x_1-x_2)+\dots+a_n x_{n-1}]^{\alpha_1+\dots+\alpha_n}}$$

We can rewrite the α β γ integral

$$I^{(2)}_{\alpha\beta\gamma} = \frac{\Gamma(\alpha+\gamma)}{\Gamma(\alpha)\Gamma(\gamma)} \int_0^1 dx \int d^4k_1 \int d^4k_2 \frac{(1-x)^{\alpha-1}x^{\gamma-1}}{[k_1^2+x(1-x)(k_2-p)^2+M_x^2]^{\alpha+\gamma}(k_2^2+M_2^2)^\beta}$$

Generalising to 2-loops and higher

To get this integral in sufficient ILI form, we reparametrise the divergence and transform it to u-space (i.e., UV divergence preserving parameterisation. In the present case, we can use the formula

$$\frac{1}{a^\alpha b^\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 du \; \frac{u^{\beta-1}}{[a+bu]^{\alpha+\beta}}$$

Wu, 2002

After some work, the 2-loop scalar-like integral takes the form

$$I^{(2)}_{\alpha\beta\gamma} = \frac{\Gamma(\alpha+\beta+\gamma-2)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int_0^1 dx \; x^{\gamma-1}(1-x)^{\alpha-1}(q_0^2)^{\alpha+\gamma-2} \sum_{i=0}^{\beta-1} c_i^{\beta-1}(x(1-x)q_0^2)^i$$
$$\int \frac{d^4\hat{k}_1}{\hat{k}_1^2} \frac{1}{(\hat{k}_1^2 + x(1-x)q_0^2)^{\alpha+\gamma-1+i}} \int d^4 k_2 \; \frac{1}{(k_2^2 + M_2^2 + \mu^2_{\hat{k}_1^2})^{\alpha+\beta+\gamma-2}},$$

in which it is noticed that the final k_2 integral is a one-fold ILI for the two-loop graphs. It follows now that we can regularise, similarly as in the 1-loop case.

To give a quick example of what the regularised integral looks like in this case, upon introducing a smooth cutoff and using change of variables we have

$$I^{(2)R}_{\alpha\beta\gamma} = \frac{\Gamma(\alpha+\beta+\gamma-2)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int_0^1 dx\; x^{\gamma-1}(1-x)^{\alpha-1}(q_0^2)^{\alpha+\gamma-2} \sum_{i=0}^{\beta-1} c_i^{\beta-1}(x(1-x)q_0^2)^i$$

$$\frac{1}{\hat{\Lambda}_1^3} \int \frac{d^4\hat{x}_1}{\hat{x}_1^2} \frac{1}{(\hat{x}_1^2 + x(1-x)q_0^2)^{\alpha+\gamma-1+i}} \eta(\hat{x}_1)$$

$$\frac{1}{\Lambda_2} \int d^4k_2 \, \frac{1}{(x_2^2 + M_2^2/\Lambda_2 + \mu^2_{\hat{k}_1^2})^{\alpha+\beta+\gamma-2}} \eta(x_2)$$

Note that depending on the value of α,β,γ the integrals will be logarithmically or quadratically divergent.

n-fold ILIs

Much more generally, we can define n-fold ILIs. For example, one form of the scalar-like n-fold ILI is given as

$$I^{(n)}_{\alpha_i \alpha_{ij}} = \int d^4 k_n \frac{1}{(k_n^2 + M_n^2)^{\alpha_n}} \prod_{j>i} \prod_{i=1}^{n-1} \int d^4 k_i \frac{1}{(k_i^2 + M_i^2)^{\alpha_i}} \frac{1}{[(k_i - k_j + p_{ij})^2 + M_{ij}^2]^{\alpha_{ij}}}$$

Wu, 2003

We can similarly write tensor-like n-fold ILIs. Upon use of UVDP parameterisation we have for both cases

$$I^{(n)}_{\Delta} = \prod_{i=1}^{n-1} \int_0^\infty du_i \frac{F_{is}(x_{lm})}{(u_i + \rho_i)^{\Delta_{is}}} I^{(1)}_{\Delta_n}(\mu_n^2)$$

$$I^{(n)}_{\Delta \mu\nu} = \prod_{i=1}^{n-1} \int_0^\infty du_i \frac{F_{is}(x_{lm})}{(u_i + \rho_i)^{\Delta_{is}}} I^{(1)}_{\Delta_n \mu\nu}(\mu_n^2),$$

where

$$I^{(1)}_{\Delta_n}(\mu_n^2) = \int d^4 k_n \frac{1}{(k_n^2 + M^2 + \mu_n^2)^{\Delta_n}}$$

$$I^{(1)}_{\Delta_n \mu\nu}(\mu_n^2) = \int d^4 k_n \frac{k_{n\mu} k_{n\nu}}{(k_n^2 + M^2 + \mu_n^2)^{\Delta_n + 1}}$$

Then the regularisation proceeds similarly to what was discussed before.

QFTs: comments, ongoing work, and future considerations

To summarise, here is list of ongoing research topics as well as future considerations.

- Gauge invariance and master equation at arbitrary loop order
- Discrete / position space picture
- Chiral, topological and supersymmetric theories
- Anomalies
- Cutkosky rules
  - Unitarity, locality, and causality
- Yang-Mills (all-plus helicity)
  - An interesting example.
- Gauged Nambu-Jona-Lasinio model
  - E.g. Dimensional regularisation cannot obtain the correct gap equation because the quadratic divergent term is destroyed.
    - It is possible to restrict η such that the quadratic divergent term is preserved. An interesting test.
- Schwinger representation of η-class regulators

Ongoing: Stringy η-class regulators

An early motivation of this research was the observation that natural choices of η are typically multi-term exponentially decreasing functions (very stringy).

Two primary features we can attribute to the finiteness of string amplitudes: 1) modular invariance and 2) unique behaviour of the worldsheet Green function at short distances.

- If this very general notion of η-regularisation has a chance to capture the important qualities of finite string amplitudes, it will need in some way to play a role with this behaviour in the UV.

There is a longstanding conjecture that in the appropriate limit the sum of QFT amplitudes can be connected exactly to the sum of string theory amplitudes. Schematically,

$$\sum_{\gamma} \frac{1}{|Aut\ \gamma|} \int_{Met(\gamma)} \int_{\gamma \to \mathbb{R}^n} \text{some QFT} \longleftrightarrow \sum_{g} \int_{Mg} \int_{\Sigma \to \mathbb{R}^n} \text{some ST}$$

Klebanov, 1991;
Schubert, 2001;
Gopakumar, 2003;
and many others

This is just like the story in gauge-gravity duality in which the large N limit of field theory is some string theory.

However, constructing this map is difficult.

- It is furthermore also difficult to, in some sense, consider extracting and preserving the regulating features in the UV as some stringy η and then map to QFT.

Ongoing: Stringy η-class regulators

Encouragingly, we already have some examples we can identify as stringy η's, or closely related.

- For instance, a preliminary form of a modular invariant regulator based on [Abel and Dienes, 2021].
  - Important to study more about how it behaves, etc. particular in moduli space.

One difficultly is that stringy η's are different than field theory η's (naturally much more general).

- This in part has to do with modular invariance and the different ways the modular parameters are seen.
  - One thing we need to understand more is what happens to the modular parameters when, for instance, considering schematically QFT η <--> Stringy η.

  - For instance, consider the trivial and well-known story from the string partition function and comparing with infinite particle spectrum (after implementing level matching condition etc.):

Modular invariance forces string partition function to the fundamental domain

ST

$$Re(\tau) \in [-\frac{1}{2}, \frac{1}{2}]$$

$$Im(\tau) : |\tau| \geq 1$$

QFT

$$Re(\tau) \in [-\frac{1}{2}, \frac{1}{2}]$$

$$Im(\tau) \in [0, \infty)$$

Tong, 2009

Ongoing: Stringy η-class regulators

One way to probe some of these questions is in the proper-time representation and trying to map to the worldline formalism, where the proper time τ in some sense can be seen as the modular parameter QFT is most closely related to string theory.

$$(k^2+m^2)^{-1} = \int_0^\infty d\tau\ e^{-\tau(k^2+m^2)}$$

Sen et al., 2017

Very crudely, the parameters in string theory labelling the moduli space of Riemann surfaces can be seen to play the role of the Schwinger parameters.

- We can generalising η-regularisation to the Schwinger representation.
    - A possible next step would be to explore worldline picture as related to string amplitudes (as discussed on next slide).

Ongoing: Stringy η-class regulators

There are a number of other potentially insightful ways to probe the exponential damping behaviour of string theory in the UV, in which (in a preliminary way) we can "pick off" stringy η's and study them.

- [Abel and Dondi, 2019; Abel and Lewis, 2020;]: in the Gross-Mende regime these exponential damped amplitudes can be seen quite explicitly. This exponential damping, which comes about as the exponential correction in the Green function (lowest mode in the tower of states) is associated with a saddle point in moduli space.
    - Existence of saddle point in moduli space is a signature of stringy behaviour. Interesting to study and compare in relation to constraining behaviour of η-class regulators in moduli space
- [Skliros and Lust, 2019]: handle operator formulation of string amplitudes
    - Highly technical; handle operator insertions allow one to represent higher genus string amplitudes as lower-genus amplitudes.
        - All loop amplitudes may be reduced to sphere amplitudes.
        - Recover the Feynman propagator from the moduli integral of the three-point amplitudes?
            - Interesting to ask about the modular group, and whether the lovely behaviour of string amplitudes in the UV can be preserved as an artifact (again, related to saddle point behaviour).
- [Chiaffrino and Sachs, 2021; Erbin, 2021] In string field theory propagators found dressed with an exponential decreasing term.
        - Part of the contribution to the exponential comes from adding "stubs", which can also be done for point-particle QFT.
    - In momentum space, this amounts to adding a cutoff that resembles a stringy η.

Concluding comments

- We have briefly described a new generalised, symmetry preserving regularisation prescription.
  - η-regularisation seems to capture all other common regularisation schemes, as well as a number of less common generalised prescriptions.
  - It currently resembles what one would anticipate of a 'master regularisation'.

- In addition to string theory and QFT, there are a lot of interesting results in analytic number theory that we are still trying to make sense of!
  - Work is ongoing to better understand numerous sum and integral identities.
  - Examples: further study of η-regularisation and its relation to analytic continuation.
    - Modular forms
    - Resurgence theory

Thanks for listening!